PhD Final Oral Defense

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Presentation Outline

- Introduction & Motivation
- Related Research & Notation
- Fundamental Problem Statement & Solution
- Application Examples
- Summary & Future Research

Introduction & Motivation

Dissertation Topic:

IMPLEMENTABLE NEAR-OPTIMAL DIRECT
SAMPLED-DATA CONTROLLER SYNTHESIS
FOR INFINITE-DIMENSIONAL SYSTEMS

Areas of Application

- Industrial Control Processes many of which are formulated as distributed parameter control problems:
  - Chemical plants
  - Steel making plants
  - Semiconductor manufacturing
  - Environmental/Energy management control

- Space Structures and Communication
  - Large and multiple aperture telescopes
  - Lightweight shape maintaining antennas
  - Large flexible structures
  - Acoustic structural control

Infinite-dimensional Sampled-Data Setting Benefits

Microprocessor Control Element Low internal noise and time stable; Complex control algorithms readily implemented; Control algorithms easily reconfigured; Self checking/built-in-test capable

Distributed Parameter Systems Accurate description of some systems which is required for satisfying stringent performance requirements

$\mathcal{H}^\infty$-Control A useful framework for analyzing and synthesizing controllers which satisfy nominal and robust performance requirements

Direct Sampled-Data Synthesis Demonstrated benefits over indirect approaches; easier to trade desired performance against required sample rate

Finite-Dimensional Controller Required for engineering implementation

Typical Engineering Practice

- Approximate/Design

  Method 1: Approximate infinite-dimensional system; Design controller using continuous-time methods; Discretize controller for sampled-data implementation

  Method 2: Approximate infinite-dimensional system; Discretize approximant model; Design controller using discrete-time methods

- Shortcomings of Typical Engineering Practice: Degradation in obtainable system performance due to heuristic model approximation and approximated sampled-data design
  - Excludes intersample behavior
  - High sample rates
  - No performance guarantees

Research Objective

Design of near-optimal finite dimensional sampled-data controllers for infinite-dimensional systems

- Intersample signal behavior included
- Systematic design method w/guaranteed performance bounds
- Applicable to a large class of DPS

Related Research

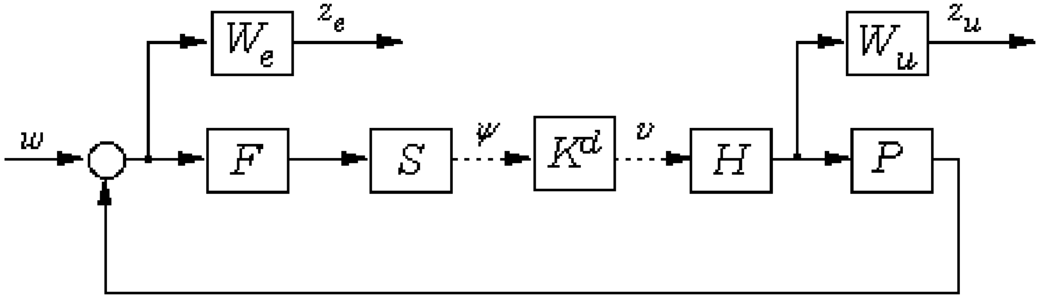
- A *Design/Approximate (Direct)* approach in which a controller is designed using infinite-dimensional direct sampled-data techniques. If the resulting discrete-time controller is infinite-dimensional, a finite-dimensional approximant is obtained.

- This approach is not readily amenable to control synthesis design via computer aided design packages and requires a high level of mathematical sophistication.

Research Contributions

- Systematic method for designing finite-dimensional sampled-data controllers for a large class of infinite-dimensional systems
- Method based on finite-dimensional approximants
- Finite-dimensional technique for determining infinite-dimensional optimal performance
- Intersample behavior included in process
- A priori computable approximant order
- Guaranteed performance bounds

Infinite-Dimensional Sampled Data System Configuration



F	strictly causal anti-aliasing filter in $\mathbb{R}\mathcal{H}^\infty(\mathbb{C}_+)$
P	infinite-dimensional plant
W_e	sensitivity weighting filter
W_u	control sensitivity weighting filter
K^d	discrete-time controller, possibly infinite-dimensional
H	zero-order hold device
S	sampler synchronized with H

Sampled-Data Mixed-Sensitivity Performance Measure

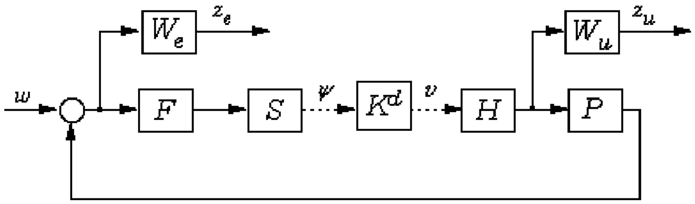
Definition. *Suppose* $G, M \in \boldsymbol{\mathcal{L}}^{\boldsymbol{e}}\left(\mathcal{L}^{2e}\right)$, $W_e, W_u, F, \in \boldsymbol{\mathcal{L}}\left(\mathcal{L}^2\right)$, *and* $V \in \boldsymbol{\mathcal{L}}\left(\ell^2\right)$ *are causal and LTI such that* $HK^d(G,V)SF$ *internally stabilizes* M*. The* mixed-sensitivity *of the pair* $(M, HK^d(G,V)SF)$*, denoted* J_{mix}*, is defined as the map* $J_{\mathrm{mix}}(\cdot,\cdot) : \boldsymbol{\mathcal{L}}^{\boldsymbol{e}}\left(\mathcal{L}^{2e}\right) \times \boldsymbol{\mathcal{L}}^{\boldsymbol{e}}\left(\mathcal{L}^{2e}\right) \times \boldsymbol{\mathcal{L}}\left(\ell^2\right) \to \mathbb{R}_+$ *where*

$$J_{\mathrm{mix}}(M, K^d(G,V)) \overset{\mathrm{def}}{=} \left\| \begin{pmatrix} W_e \\ W_u HK^d(G,V)SF \end{pmatrix} (I - MHK^d(G,V)SF)^{-1} \right\|_{\mathcal{L}^2 \to \mathcal{L}}$$

Sampled-Data Performance Measure Definitions

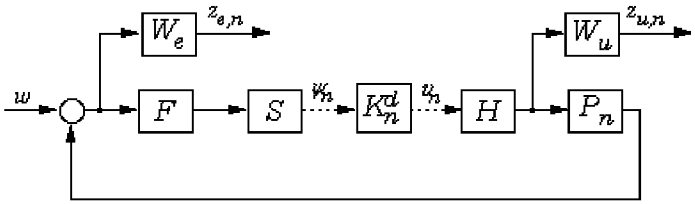
Optimal Performance

$$\mu_{opt} \overset{\text{def}}{=} \inf_{Q^d \in \mathcal{Z}^{-1}[\mathcal{H}^\infty(\mathbb{D})]} J_{\text{mix}}(P, K^d(P, Q^d))$$



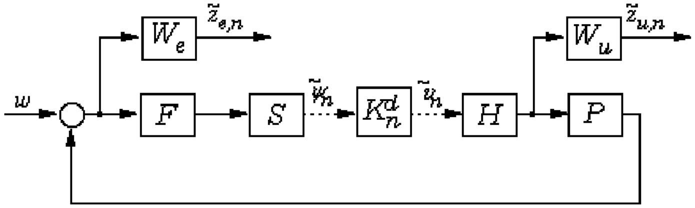
Expected Performance

$$\mu_n \overset{\text{def}}{=} \inf_{Q^d \in \mathcal{Z}^{-1}[\mathbb{R}\mathcal{H}^\infty(\mathbb{D})]} J_{\text{mix}}(P_n, K^d(P_n, Q^d))$$



Actual Performance

$$\tilde{\mu}_n \overset{\text{def}}{=} J_{\text{mix}}(P, K_n^d)$$



Problem Statements

Approximate/Design Find conditions on performance measure and system approximants such that

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}$$

Purely Finite Dimensional Find conditions on performance measure and system approximants such that

$$\lim_{n\to\infty} \mu_n = \mu_{opt}$$

Areas Utilized for Technical Solution

- $\mathcal{H}^\infty$-Optimal Sample-Data Control of Finite-Dimensional Systems
- Lifting of Continuous-Time System
- $\mathcal{H}^\infty$ Finite-Dimensional Controller Synthesis for Continuous-Time Infinite-Dimensional Systems
- Well Behaved Performance Measures
- Approximation Methods

Lift Operator

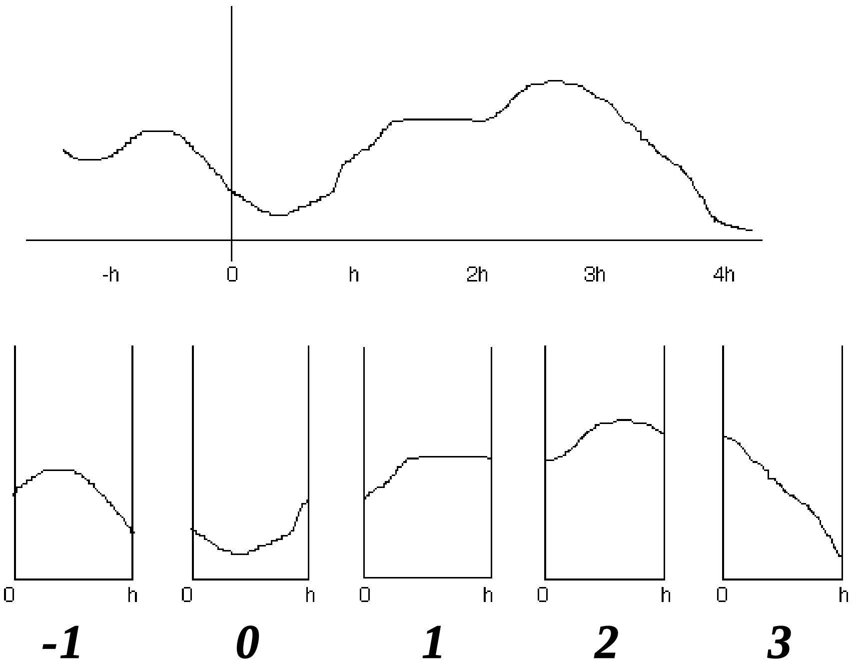
- Lift operator is an isomorphism between Hilbert spaces, i.e.

$$L^2(\quad) \equiv L^2(\quad, \quad^n) \to \ell^2(Z,K)$$

$$K \equiv L^2([0,h), \quad^n) \equiv L^2[0,h)$$

- Maps continuous-time signals into discrete-time signals
- Preserves all standard algebraic and feedback interconnection operations
- Feedback stability preserved

Action of Lift Operator



Induced Norm Equivalence

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$$\|T\| = \sup_{\|w\|_{L^2}=1} \|z\|_{L^2}$$

$$\|\underline{T}\| = \sup_{\|\underline{w}\|_{\ell^2}=1} \|\underline{z}\|_{\ell^2}$$

$$= \|\hat{\underline{t}}\|_{H^\infty} = \sup_{0\le\theta\le 2\pi} \left\|\hat{\underline{t}}\left(e^{\theta}\right)\right\|$$

$$= \|T\|$$

$$\|T_{eq,d}\| = \sup_{\|\omega\|_{\ell^2}=1} \|\zeta\|_{\ell^2}$$

$$= \|\hat{t}_{eq,d}\|_{H^\infty}$$

$$= \|T\|$$

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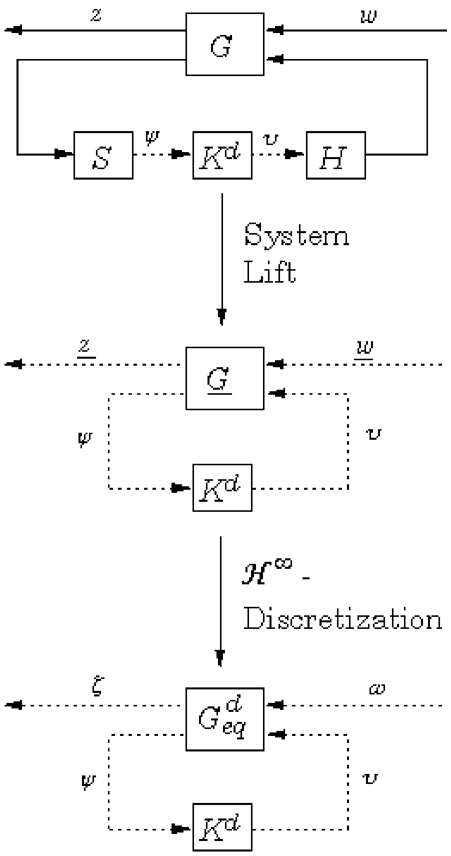
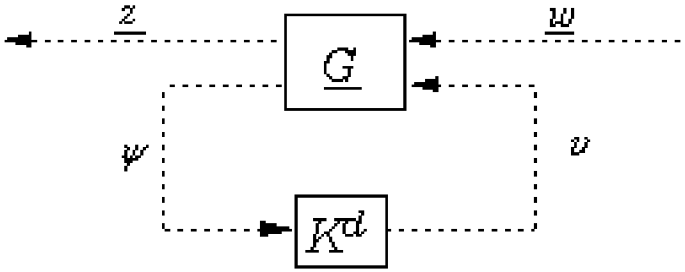


Lifted Sampled-Data Feedback Loop in Standard Form



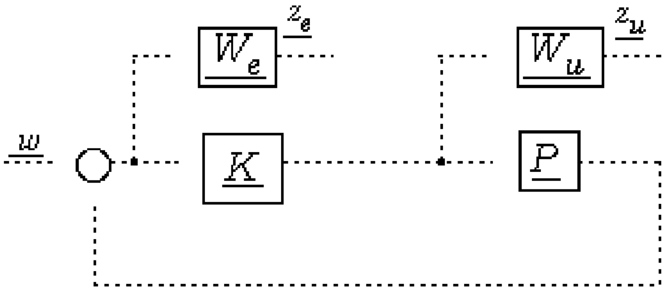
where

$$\underline{G} = \left[\begin{array}{cc} L\begin{bmatrix} W_e \\ 0 \end{bmatrix} L^{-1} & L\begin{bmatrix} W_eP \\ W_u \end{bmatrix} H \\ SFL^{-1} & SFPH \end{array}\right].$$

Lifted Sampled-Data Feedback Loops

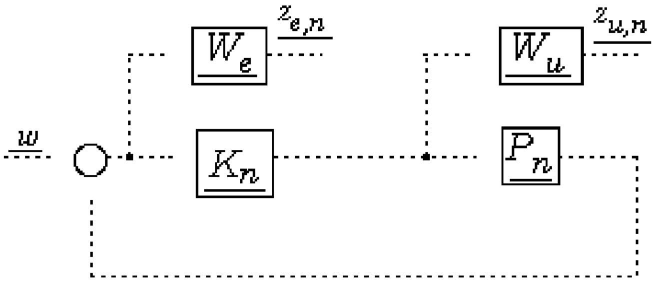
Optimal Performance

$$\mu_{opt} = \inf_{\underline{\hat{Q}} \in \underline{\mathcal{H}}^{\infty}(\mathbb{D})} \left\| \begin{pmatrix} \underline{\hat{W}_e} \\ \underline{\hat{W}_u}\,\underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}}) \end{pmatrix} \left(I - \underline{\hat{P}}\,\underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}})\right)^{-1} \right\|_{\mathcal{H}^{\infty}}$$



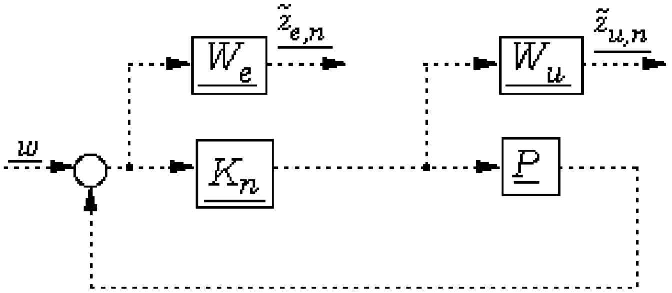
Expected Performance

$$\mu_n = \inf_{\underline{\hat{Q}} \in \underline{\mathbb{R}\mathcal{H}}^{\infty}(\mathbb{D})} \left\| \begin{pmatrix} \underline{\hat{W}_e} \\ \underline{\hat{W}_u}\,\underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}}) \end{pmatrix} \left(I - \underline{\hat{P}_n}\,\underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}})\right)^{-1} \right\|_{\mathcal{H}^{\infty}}$$



Actual Performance

$$\tilde{\mu}_n = \left\| \begin{pmatrix} \underline{\hat{W}_e} \\ \underline{\hat{W}_u}\,\underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}_n}) \end{pmatrix} \left(I - \underline{\hat{P}}\,\underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}_n})\right)^{-1} \right\|_{\mathcal{H}^{\infty}}$$



Sampled-Data System Lifting

Lift sampled-data system operators:

$$\begin{aligned}
\underline{W_e} &= L W_e L^{-1} \\
\underline{W_u} &= L W_u L^{-1} \\
\underline{P} &= L P L^{-1} \\
\underline{K} &= L K L^{-1},\ K \overset{\text{def}}{=} H K^d S F \\
\underline{P_n} &= L P_n L^{-1} \\
\underline{K_n} &= L K_n L^{-1},\ K_n \overset{\text{def}}{=} H K_n^d S F
\end{aligned}$$

Lifted Stabilizing Controllers

The set of lifted discrete-time controllers which internally stabilize $\underline{\hat{P}}$ is

$$S(\underline{\hat{P}}) \overset{\text{def}}{=} \{\underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}})) \overset{\text{def}}{=} (\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}})(\underline{\hat{D}_k} - \underline{\hat{N}_p}\underline{\hat{Q}})^{-1}\underline{\hat{F}} \mid \underline{\hat{Q}}(Q^d) \in \underline{\mathcal{H}}^\infty(\mathbb{D})\,\}$$

where

$$\underline{\hat{\tilde{N}}_p}\underline{\hat{N}_k} + \underline{\hat{\tilde{D}}_p}\underline{\hat{D}_k} = L\widehat{HSL}^{-1} \overset{\text{def}}{=} \underline{\widehat{HS}},$$

$$\mathcal{Q} : \mathcal{H}^\infty(\mathbb{D}) \to \underline{\mathcal{H}}^\infty(\mathbb{D}) \overset{\text{def}}{=} \mathcal{Z}[LH\mathcal{Z}^{-1}[\mathcal{H}^\infty(\mathbb{D})]SL^{-1}] \subset \boldsymbol{\mathcal{H}}^{\boldsymbol{\infty}}$$

and

$$\mathcal{Q}(\hat{Q}^d) \quad = \quad LH\widehat{\mathcal{Z}^{-1}[\hat{Q}^d]}SL^{-1} \overset{\text{def}}{=} \underline{\hat{Q}}(Q^d)$$

$$\begin{aligned}
\mathcal{Q}(\hat{N}_k^d) &= LH\widehat{\mathcal{Z}^{-1}[\hat{N}_k^d]}SL^{-1} \overset{\text{def}}{=} \underline{\hat{N}_k} \\
\mathcal{Q}(\hat{D}_k^d) &= LH\widehat{\mathcal{Z}^{-1}[\hat{D}_k^d]}SL^{-1} \overset{\text{def}}{=} \underline{\hat{D}_k} \\
\mathcal{Q}(\hat{N}_p^d) &= LH\widehat{\mathcal{Z}^{-1}[\hat{N}_p^d]}SL^{-1} \overset{\text{def}}{=} \underline{\hat{N}_p} \\
\mathcal{Q}(\hat{D}_p^d) &= LH\widehat{\mathcal{Z}^{-1}[\hat{D}_p^d]}SL^{-1} \overset{\text{def}}{=} \underline{\hat{D}_p} \\
\mathcal{Q}(\hat{\tilde{N}}_p^d) &= LH\widehat{\mathcal{Z}^{-1}[\hat{\tilde{N}}_p^d]}SL^{-1} \overset{\text{def}}{=} \underline{\hat{\tilde{N}}_p} \\
\mathcal{Q}(\hat{\tilde{D}}_p^d) &= LH\widehat{\mathcal{Z}^{-1}[\hat{\tilde{D}}_p^d]}SL^{-1} \overset{\text{def}}{=} \underline{\hat{\tilde{D}}_p}.
\end{aligned}$$

The operator $\underline{\hat{K}} \overset{\text{def}}{=} \underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}}) = \underline{\hat{Q}}\,\underline{\hat{F}}(I - \underline{\hat{P}}\,\underline{\hat{Q}}\,\underline{\hat{F}})^{-1} \overset{\text{def}}{=} \underline{\hat{Q}^F}(I - \underline{\hat{P}}\,\underline{\hat{Q}^F})^{-1}$ internally stabilizes the *stable* infinite-dimensional operator $\underline{\hat{P}}$.

Performance Measure Affine Transformation

Given that $\underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}}) \overset{\text{def}}{=} -\underline{\hat{Q}}(I - \underline{\hat{P}}\underline{\hat{Q}})^{-1}$, the *optimal performance* and *expected performance* become, respectively

$$\mu_{opt} = \inf_{\underline{\hat{Q}} \in \underline{\mathcal{H}}^{\infty}(\mathbb{D})} \left\| \begin{array}{c} \underline{\hat{W}_e}(I - \underline{\hat{P}}\underline{\hat{Q}}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}} \end{array} \right\|_{\boldsymbol{\mathcal{H}}^{\infty}}$$

$$\mu_{n} = \inf_{\underline{\hat{Q}} \in \underline{\mathbb{R}\mathcal{H}}^{\infty}(\mathbb{D})} \left\| \begin{array}{c} \underline{\hat{W}_e}(I - \underline{\hat{P}_n}\,\underline{\hat{Q}}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}} \end{array} \right\|_{\boldsymbol{\mathcal{H}}^{\infty}}$$

Problem Solutions

Assumption 1. $\hat{W}_e, \hat{W}_u, \hat{W}_u^{-1} \in \mathbb{R}\mathcal{H}^\infty(\mathbb{C}_+)$

Construction (Finite-Dimensional Approximants: $\{\hat{P}_n\}_{n=1}^\infty$) Let $\{\hat{P}_n\}_{n=1}^\infty$ denote a sequence of $\mathbb{R}\mathcal{H}^\infty(\mathbb{C}_+)$ matrix-valued functions such that

$$\lim_{n\to\infty} \|\hat{P}_n - \hat{P}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} = 0.$$

More specifically, suppose that one chooses a desired performance tolerance $\varepsilon_d > 0$, however small. Let $\varepsilon \in [0, 1)$ satisfy the inequality

$$\varepsilon \le \frac{\varepsilon_d}{\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} + 3 + \varepsilon_d}.$$

Define the (*a priori* known) quantity

$$B \overset{\text{def}}{=} B(\varepsilon, \hat{W}_e, \hat{W}_u) \overset{\text{def}}{=} \|\hat{W}_u^{-1}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} \; (\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} + \varepsilon).$$

Given this, choose $N \in \mathbb{Z}_+$ such that

$$\|\hat{P}_n - \hat{P}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} < \delta \overset{\text{def}}{=} \delta(\varepsilon, \hat{W}_e, \hat{W}_u) \overset{\text{def}}{=} \min\left\{ \frac{\varepsilon}{\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} B}, \frac{\varepsilon}{B} \right\} \quad (1)$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$.

Half-Plane Algebra Isometric Isomorphism

Proposition 1. *For every* $\hat{P} \in \mathcal{A}_{\mathbb{R}}$*, there exists a unique* $\underline{\hat{P}} \in \mathcal{A}$ *such that* $Z^{-1}\Theta_{\underline{\hat{P}}}Z = \underline{P}$*. Moreover,* $\|\hat{P}\|_{\mathcal{H}^\infty} = \|\underline{\hat{P}}\|_{\boldsymbol{\mathcal{H}}^\infty}$

Lifted System Parameter Properties

Lemma 1. *The lifted weighting filters*

$$\underline{\hat{W}_e}, \underline{\hat{W}_u}, \underline{\hat{W}_u}^{-1} \in \mathcal{A}.$$

The lifted approximants $\underline{\hat{P}_n}$ *and the lifted plant* $\underline{\hat{P}}$ *satisfy*

$$\|\underline{\hat{P}_n} - \underline{\hat{P}}\|_{\boldsymbol{\mathcal{H}}^\infty} < \delta$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$.

Problem Solutions: Upper-semicontinuity

Let $\mathcal{Q}(\hat{Q}_o^d) = \underline{\hat{Q}_o} \in \underline{\mathcal{H}}^{\infty}(\mathbb{D})$ satisfy the following inequality:

$$\left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}}\,\underline{\hat{Q}_o}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_o} \end{matrix} \right\|_{\mathcal{H}^{\infty}} \leq \mu_{opt} + \varepsilon \tag{2}$$

Upper-semicontinuity

Proposition 2. *Given Assumption 1, it follows that*

$$\|\underline{\hat{Q}_o}\|_{\mathcal{H}^\infty} \leq B \tag{3}$$

and

$$\mu_n \leq \mu_{opt} + 2\varepsilon \tag{4}$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. *Moreover,*

$$\lim_{n\to\infty} \inf_{k\geq n} \mu_k \leq \lim_{n\to\infty} \sup_{k\geq n} \mu_k \leq \mu_{opt} \tag{5}$$

Upper-semicontinuity

Proof. Since $\underline{\hat{W}_u}^{-1} \in \boldsymbol{\mathcal{H}}^{\infty}$ and $\mu_{opt} \leq \|\underline{\hat{W}_e}\|_{\boldsymbol{\mathcal{H}}^{\infty}}$, it follows that

$$\begin{aligned} \|\underline{\hat{Q}_o}\|_{\boldsymbol{\mathcal{H}}^{\infty}} &\leq \|\underline{\hat{W}_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}} \|\underline{\hat{W}_u}\,\underline{\hat{Q}_o}\|_{\boldsymbol{\mathcal{H}}^{\infty}} \leq \|\underline{\hat{W}_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}} (\mu_{opt} + \varepsilon) \\ &\leq \|\underline{W_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}} \left(\|\underline{\hat{W}_e}\|_{\boldsymbol{\mathcal{H}}^{\infty}} + \varepsilon \right) \end{aligned} \tag{6}$$

However,

$$\|\underline{W_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}} (\|\underline{\hat{W}_e}\|_{\boldsymbol{\mathcal{H}}^{\infty}} + \varepsilon) = \|\hat{W}_u^{-1}\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)} (\|\hat{W}_e\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)} + \varepsilon) \overset{\text{def}}{=} B \tag{7}$$

One should note that to obtain this bound, the invertibility condition on

$\hat{W}_u$ was critical. Now, consider the following inequality:

$$\begin{aligned}\mu_n &\le \left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}_n}\underline{\hat{Q}_o}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_o} \end{matrix} \right\|_{\mathcal{H}^\infty} \\ &\le \left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}}\underline{\hat{Q}_o}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_o} \end{matrix} \right\|_{\mathcal{H}^\infty} + \left\| \underline{\hat{W}_e}(\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_o} \right\|_{\mathcal{H}^\infty} \end{aligned} \tag{8}$$

Using the near-optimality of $\underline{\hat{Q}_o}$ [see Equation (2)] and the bound for $\underline{\hat{Q}_o}$ obtained in (3) yields

$$\mu_n \le \mu_{opt} + \varepsilon + B\|\underline{\hat{W}_e}\|_{\mathcal{H}^\infty}\|\underline{\hat{P}_n} - \underline{\hat{P}}\|_{\mathcal{H}^\infty} \tag{9}$$

The proof then follows from the construction given for $\hat{P}_n$ [see Equation (1)] and Lemma 1. □

Lower-semicontinuity

Let $\mathcal{Q}(\hat{Q}_n^d) = \underline{\hat{Q}_n} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$ satisfy the following inequality:

$$\left\| \begin{array}{c} \underline{\hat{W}_e}(I - \underline{\hat{P}_n}\underline{\hat{Q}_n}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_n} \end{array} \right\|_{\boldsymbol{\mathcal{H}}^\infty} \leq \mu_n + \varepsilon \tag{10}$$

Lower-semicontinuity

Proposition 3. *Given Assumption 1, it follows that*

$$\|\underline{\hat{Q}_n}\|_{\mathcal{H}^\infty} \leq B \tag{11}$$

for all $n \in \mathbb{Z}_+$ *and*

$$\mu_{opt} \leq \mu_n + 2\varepsilon \tag{12}$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. *Moreover,*

$$\mu_{opt} \leq \lim_{n\to\infty} \inf_{k\geq n} \mu_k \leq \lim_{n\to\infty} \sup_{k\geq n} \mu_k \tag{13}$$

Lower-semicontinuity

Proof. Since $\underline{\hat{W}_u}^{-1} \in \boldsymbol{\mathcal{H}}^{\infty}$ and $\mu_n \leq \|\underline{\hat{W}_e}\|_{\boldsymbol{\mathcal{H}}^{\infty}}$ for all $n \in \mathbb{Z}_+$, it follows that

$$\begin{aligned} \|\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^{\infty}} &\leq \|\underline{\hat{W}_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}} \|\underline{\hat{W}_u}\,\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^{\infty}} \leq \|\underline{\hat{W}_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}}(\mu_n + \varepsilon) \\ &\leq \|\underline{\hat{W}_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}}(\|\underline{\hat{W}_e}\|_{\boldsymbol{\mathcal{H}}^{\infty}} + \varepsilon) \end{aligned} \tag{14}$$

However,

$$\|\underline{\hat{W}_u}^{-1}\|_{\boldsymbol{\mathcal{H}}^{\infty}}(\|\underline{\hat{W}_e}\|_{\boldsymbol{\mathcal{H}}^{\infty}}+\varepsilon) = \|\hat{W}_u^{-1}\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)}(\|\hat{W}_e\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)}+\varepsilon) \overset{\text{def}}{=} B \tag{15}$$

To obtain this uniform bound for $\{\underline{\hat{Q}_n}\}_{n=1}^{\infty}$, the invertibility of $\hat{W}_u$ in $\mathcal{H}^{\infty}(\mathbb{C}_+)$ was, once again, the key. To complete the proof, consider the following inequality:

$$
\begin{aligned}
\mu_{opt} &\le \left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}}\,\underline{\hat{Q}_n}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_n} \end{matrix} \right\|_{\boldsymbol{\mathcal{H}}^\infty} \\
&\le \left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}_n}\,\underline{\hat{Q}_n}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_n} \end{matrix} \right\|_{\boldsymbol{\mathcal{H}}^\infty} + \left\| \underline{\hat{W}_e}(\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_n} \right\|_{\boldsymbol{\mathcal{H}}^\infty} \qquad (16)
\end{aligned}
$$

Using the near-optimality of $\underline{\hat{Q}_n}$ [see Equation (10)] and the uniform bound for $\underline{\hat{Q}_n}$ obtained in (11) yields

$$\mu_{opt} \le \mu_n + \varepsilon + B\|\underline{\hat{W}_e}\|_{\boldsymbol{\mathcal{H}}^\infty}\|\underline{\hat{P}_n} - \underline{\hat{P}}\|_{\boldsymbol{\mathcal{H}}^\infty} \qquad (17)$$

The proof then follows from the construction of $\hat{P}_n$ [see Equation (1)] and Lemma 1. $\square$

Solution to Purely Finite-Dimensional Problem

Theorem 1. *Given the mixed-sensitivity weighting function standing assumptions, it follows that*

$$|\mu_n - \mu_{opt}| \leq 2\varepsilon$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. *Moreover,*

$$\lim_{n \to \infty} \mu_n = \mu_{opt}$$

Theorem 1 Proof Outline

The proof of this theorem follows from the fact that the function $\mu(\underline{\hat{P}})$ can be shown to be continuous at $\underline{\hat{P}}$ in the uniform topology on $\boldsymbol{\mathcal{H}}^{\infty}$ when $\hat{W}_u$ is invertible in $\mathcal{H}^{\infty}(\mathbb{C}_+)$.

Stability of Actual Closed Loop Operator: $(\underline{\hat{P}}, \underline{\hat{K}_n})$

Proposition 4. *Given the mixed-sensitivity weighting function standing assumptions, it follows that*

$$\|(\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^\infty} < 1$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. *Moreover,*

$$\lim_{n\to\infty} \|(\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^\infty} = 0$$

and the operator $\underline{\hat{K}_n} \stackrel{\text{def}}{=} \underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}_n}) = -\underline{\hat{Q}_n}(I - \underline{\hat{P}_n}\,\underline{\hat{Q}_n})^{-1}$ *internally stabilizes the MIMO distributed parameter operator* $\underline{\hat{P}}$ *for all except possibly a finite number of* n.

Proposition Proof

Proof. Using the uniform bound obtained for $\underline{\hat{Q}_n}$ in (11), one obtains $\|(\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_n}\|_{\mathcal{H}^\infty} \leq B\|(\underline{\hat{P}_n} - \underline{\hat{P}})\|_{\mathcal{H}^\infty}$. The proof of this proposition then follows from the construction of $\hat{P}_n$ [see Equation (1)], Lemma 1 , and the small gain theorem. $\square$

Actual Performance

Given that $\underline{\hat{K}_n} \overset{\text{def}}{=} \underline{\hat{K}}(\underline{\hat{P}_n}, \underline{\hat{Q}_n}) \overset{\text{def}}{=} -\underline{\hat{Q}_n}(I - \underline{\hat{P}_n}\,\underline{\hat{Q}_n})^{-1}$ stabilizes $\underline{\hat{P}}$ for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$, it follows that the *actual performance*, $\tilde{\mu}_n$, is well defined and given by:

$$\tilde{\mu}_n = \left\| \begin{pmatrix} \underline{\hat{W}_e}(I - \underline{\hat{P}_n \hat{Q}_n}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_n} \end{pmatrix} \left(I - (\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_n}\right)^{-1} \right\|_{\mathcal{H}^\infty} \tag{18}$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$.

Solution to $\mathcal{H}^\infty$ Approximate/Design Mixed-Sensitivity Problem

Theorem 2. *Given the mixed-sensitivity weighting function standing assumptions, it follows that*

$$\mu_{opt} \leq \tilde{\mu}_n \leq \mu_{opt} + \varepsilon_d$$

for all $n \geq N \overset{\mathrm{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. *Moreover,*

$$\lim_{n \to \infty} \tilde{\mu}_n = \mu_{opt}.$$

Theorem 2 Proof

Proof. From Equation (18), one obtains the following inequality:

$$\tilde{\mu}_n \leq \left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}_n}\underline{\hat{Q}_n}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_n} \end{matrix} \right\|_{\mathcal{H}^\infty} \frac{1}{1 - \|(\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_n}\|_{\mathcal{H}^\infty}}$$

Since $\underline{\hat{K}_n} \overset{\text{def}}{=} -\underline{\hat{Q}_n}(I - \underline{\hat{P}_n}\,\underline{\hat{Q}_n})^{-1}$ stabilizes $\underline{\hat{P}}$ for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$, it follows that

$$\mu_{opt} \leq \tilde{\mu}_n$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. Since $\mathcal{Q}(\hat{Q}_n^d) = \underline{\hat{Q}_n} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$ satisfies the following inequality:

$$\left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}_n}\underline{\hat{Q}_n}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_n} \end{matrix} \right\|_{\boldsymbol{\mathcal{H}}^\infty} \leq \mu_n + \varepsilon,$$

it follows from Theorem 1 that

$$\left\| \begin{matrix} \underline{\hat{W}_e}(I - \underline{\hat{P}_n}\underline{\hat{Q}_n}) \\ \underline{\hat{W}_u}\,\underline{\hat{Q}_n} \end{matrix} \right\|_{\boldsymbol{\mathcal{H}}^\infty} \leq \mu_n + \varepsilon \leq \mu_{opt} + 3\varepsilon$$

for all $n \geq N \stackrel{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. This, then yields

$$\mu_{opt} \leq \tilde{\mu}_n \leq \frac{\mu_{opt} + 3\varepsilon}{1 - \|(\underline{\hat{P}_n} - \underline{\hat{P}})\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^\infty}}$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. Using the uniform bound for $\underline{\hat{Q}_n}$, $\|\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^\infty} \leq B$ for all $n \in \mathbb{Z}_+$, yields

$$\mu_{opt} \leq \tilde{\mu}_n \leq \frac{\mu_{opt} + 3\varepsilon}{1 - B\|(\underline{\hat{P}_n} - \underline{\hat{P}})\|_{\boldsymbol{\mathcal{H}}^\infty}}$$

for all $n \geq N \overset{\text{def}}{=} N(\varepsilon, \hat{W}_e, \hat{W}_u)$. The proof of the theorem then follows from the the construction of $\hat{P}_n$, Lemma 1 and

$$\begin{aligned}
\frac{\mu_{opt} + 3\varepsilon}{1 - \varepsilon} &\leq \mu_{opt} + (\mu_{opt} + 3)\frac{\varepsilon}{1 - \varepsilon} \\
&\leq \mu_{opt} + (\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} + 3)\frac{\varepsilon}{1 - \varepsilon} \\
&\leq \mu_{opt} + \varepsilon_d.
\end{aligned}$$

□

Design Methodology

1. Start with the sampled-data system with infinite-dimensional plant, P, and performance measure, μ_{opt}, which takes into account intersample behavior. A specified performance criterion is that the actual performance be near-optimal. The actual performance is defined as the performance achieved from the sampled-data system using the infinite-dimensional plant and a finite-dimensional discrete-time controller, K_n^d.

2. Approximate the infinite-dimensional plant with finite-dimensional $\mathbb{R}\mathcal{H}^\infty(\mathbb{C}_+)$ approximants of *a priori* determinable order based on ε_d and the weighting filters. Use this approximant in place of the infinite-dimensional plant in the sampled-data set-up.

3. Lift the resultant finite-dimensional sampled-data system (Figure 1).

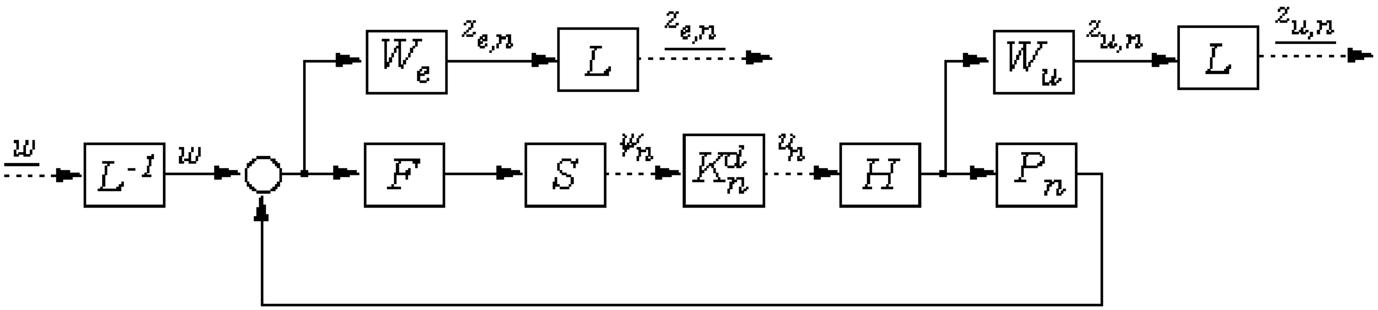


Figure 1: Sampled-Data Feedback Loop Input/Output Lifting

4. In Figure 1, $\mathcal{H}^\infty$-discretize the operator which maps $\left[\begin{array}{c} \underline{w} \\ v_n \end{array}\right] \to \left[\begin{array}{c} \underline{z_n} \\ \psi_n \end{array}\right]$.

5. Synthesize a finite-dimensional discrete-time controller, K_n^d, using "natural" discrete-time $\mathcal{H}^\infty(\mathbb{D})$ design algorithms based on the discretized finite-dimensional operator.

6. The synthesized finite-dimensional discrete-time controller results in

stable closed loop performance for the original infinite-dimensional sampled-data system with guaranteed performance.

Numerical Example

Infinite-Dimensional Plant

$$\hat{P}(s) = \frac{e^{-s}}{s+1}$$

Sample Rate

$$T_s = 0.3$$

Anti-Aliasing Filter

$$\hat{F}(s) = \frac{1}{\frac{0.3}{\pi}s + 1}$$

Weighting Filters

Error and Control Weighting Filters

$$\begin{aligned}\hat{W}_e(s) &= \frac{1}{\left(\frac{5}{2\pi}s+1\right)^2}\\ \hat{W}_u(s) &= \frac{\left(\frac{5}{2\pi}s+10^{-3}\right)}{\left(\frac{\pi}{75}s+1\right)}.\end{aligned}$$

Weighting Filter $\mathcal{H}^\infty$ Norms

$$\begin{aligned}\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} &= 1\\ \|\hat{W}_u\|_{\mathcal{H}^\infty(\mathbb{C}_+)} &= 19\end{aligned}$$

Plant Approximants

Uniform Plant Approximants

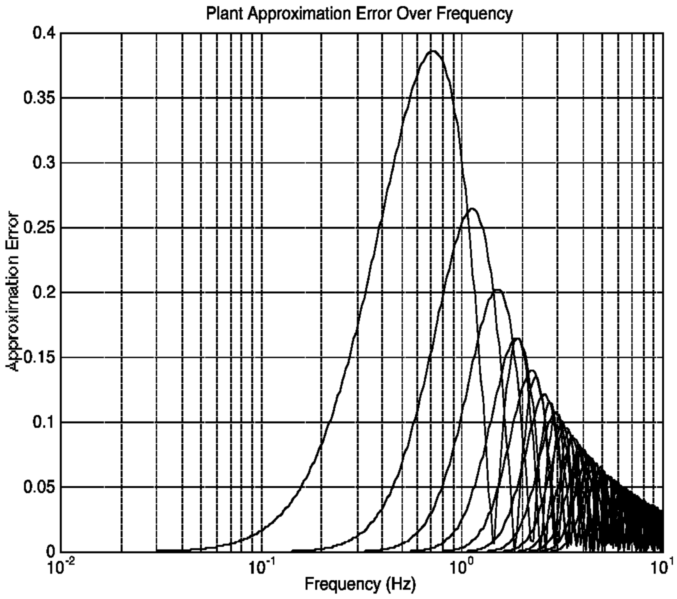
$$\hat{P}_n(s) = \frac{\hat{N}_{\mathrm{p}n}(s)}{\hat{D}_{\mathrm{p}n}(s)} \left(\frac{1}{s+1} \right)$$

$\hat{P}_n$ **uniformly approximates** $\hat{P}$

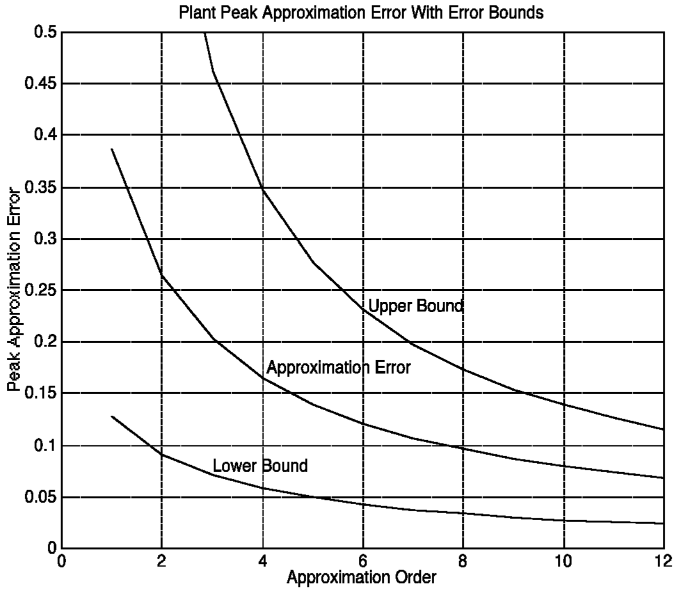
$$\lim_{n \to \infty} \|\hat{P}_n - \hat{P}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} = 0$$

Approximant Error Curves

Frequency Dependent Approximation Error

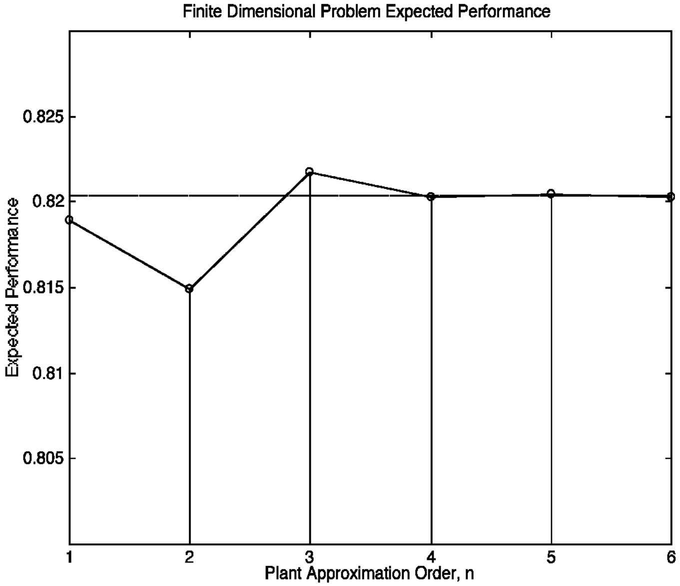


Peak Approximation Error with Error Bounds



μ_{opt} Estimation

- Convergence of expected performance, μ_n



- Indicates that infinite-dimensional sampled-data system optimal performance μ_{opt} is around 0.82

Confidence in μ_{opt} Estimation

- $|\mu_n - \mu_{opt}| \leq 2\varepsilon$ by Theorem 1
- By construction,

$$\|\hat{P}_6 - \hat{P}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} < \delta \overset{\text{def}}{=} \min\{\frac{\varepsilon}{\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} B}, \frac{\varepsilon}{B}\}, \quad \forall n \geq N(\varepsilon, \hat{W}_e, \hat{W}_u) = 6$$

where

$$\begin{aligned} B &= \|\hat{W}_u^{-1}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} \left(\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} + \varepsilon \right) \\ &= (19^{-1})\,(1+\varepsilon) \end{aligned}$$

and

$$\|\hat{P}_6 - \hat{P}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} = 0.1211$$

- The minimum ε value which satisfies these relations is

$$\varepsilon = 0.0064.$$

Given this, it follows that

$$|\mu_6 - \mu_{opt}| \leq 2\varepsilon = 0.0128.$$

- If $\mu_{opt} \approx 0.82$, then the approximation error is about 2%.

Near-Optimal Controller Synthesis

- Apply the same weighting filters

- Specify desired performance tolerance as $\varepsilon_d = 0.08$ which yields

$$\varepsilon \leq \frac{\varepsilon_d}{\|W_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} + 3 + \varepsilon_d} = \frac{0.08}{1 + 3 + 0.08} = \frac{1}{51}$$

- Applying the same plant approximants, the approximation order must satisfy

$$\|\hat{P}_n - \hat{P}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} < \delta \overset{\text{def}}{=} \min\{\frac{\varepsilon}{\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} B}, \frac{\varepsilon}{B}\}, \quad \forall n \geq N(\varepsilon, \hat{W}_e, \hat{W}_u)$$

where

$$
\begin{aligned}
B &= \|\hat{W}_u^{-1}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} \left(\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} + \varepsilon\right) \\
&= (19^{-1})\left(1 + \frac{1}{51}\right) \\
&= 53.69\text{x}10^{-3}.
\end{aligned}
$$

This results in

$$
\|\hat{P}_n - \hat{P}\|_{\mathcal{H}^\infty(\mathbb{C}_+)} \leq \frac{1/51}{53.69\text{x}10^{-3}} = 0.37.
$$

Hence, the required order must be greater than 1^{st}-order.

- Applied 2^{nd} order plant approximant

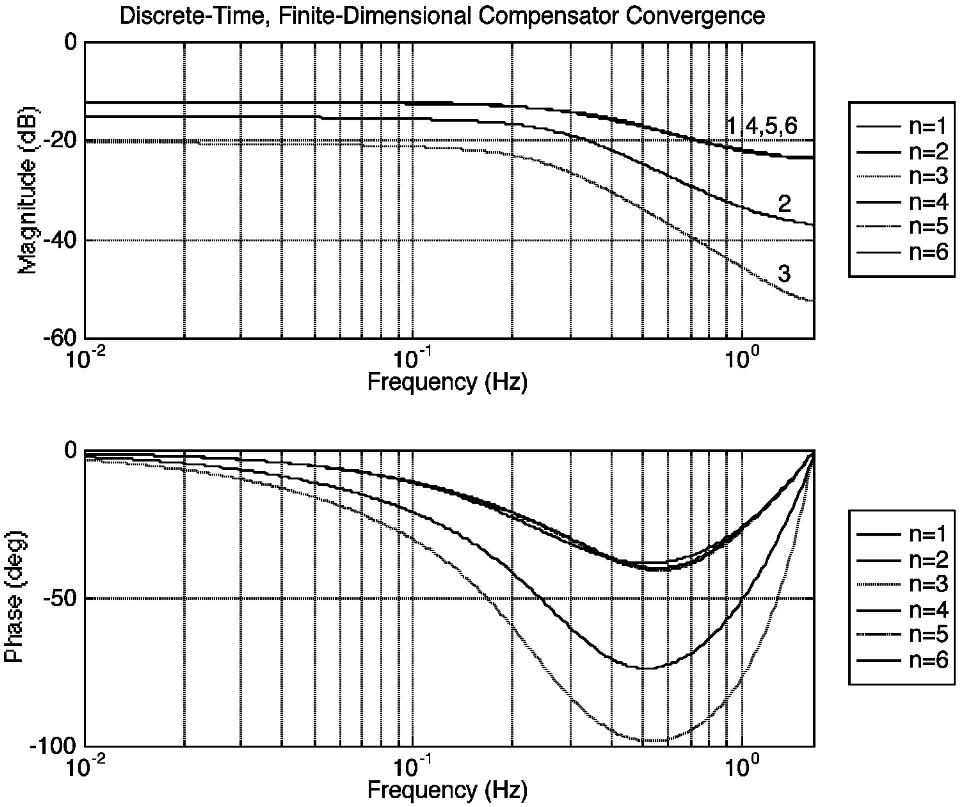
$$\hat{P}_2(s) = \frac{0.0833s^2 - 0.5s + 1}{(0.0833s^2 + 0.5s + 1)(s + 1)}$$

  yields $\mu_2 = 0.8149$.

- Theorem 1 guarantees this to be within $\pm 2\varepsilon = \pm\frac{2}{51} = 0.039$ of the optimal performance, μ_{opt}.

- By Theorem 2, the actual performance guarantee is that $0 \leq \tilde{\mu}_n - \mu_{opt} \leq 0.08$.

Discrete-Time, Finite-Dimensional Controller Convergence



Performance Measure Affine Transformation

Given that $\underline{\hat{K}}(\underline{\hat{P}}, \underline{\hat{Q}})) \overset{\text{def}}{=} (\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}})(\underline{\hat{D}_k} - \underline{\hat{N}_p}\underline{\hat{Q}})^{-1}\underline{\hat{F}}$, the *optimal performance* and *expected performance* measures become, respectively

$$\mu_{opt} = \inf_{\underline{\hat{Q}} \in \underline{\mathcal{H}}^{\infty}(\mathbb{D})} \left\| \begin{matrix} \underline{\hat{W}_e}[I - \underline{\hat{P}}(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}} \end{matrix} \right\|_{\boldsymbol{\mathcal{H}}^{\infty}}$$

$$\mu_n = \inf_{\underline{\hat{Q}} \in \underline{\mathbb{R}\mathcal{H}}^{\infty}(\mathbb{D})} \left\| \begin{matrix} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}} \end{matrix} \right\|_{\boldsymbol{\mathcal{H}}^{\infty}}$$

Continuous-Time Infinite-Dimensional Plant Assumptions

Assumption 2. *1. P has a finite number of unstable modes (Required to stabilize P by a finite dimensional controller.)*

2. $P \overset{\text{def}}{=} (\tilde{D}_p)^{-1}(\tilde{N}_p) = N_p(D_p)^{-1}$ where $N_p, D_p, \tilde{N}_p, \tilde{D}_p \in \mathcal{A}_{\mathbb{R}}$ (Double coprime factorization implies system is controllable and observable. Coprime factors are approximable by rational functions.)

3. Nonpathological sampling (Preserves system observability and controllability after the introduction of sampling. Hence, double coprime factorization exists for discretized system, $P^d \overset{\text{def}}{=} (\tilde{D}_p^d)^{-1}(\tilde{N}_p^d) = N_p^d(D_p^d)^{-1}$.)

4. Let $\tilde{D}_p, D_p \in \mathbb{R}\mathcal{H}^\infty$.

Construction of Approximants $\{P_n\}_{n=1}^{\infty}$

Construction 1. *1. Let* $\{\tilde{N}_{p_n}\}_{n=1}^{\infty} \subset \mathbb{R}\mathcal{H}^{\infty}$ *denote a sequence of functions which uniformly approximate* $\tilde{N}_p$*; i.e.*

$$\lim_{n\to\infty} \|\tilde{N}_{p_n} - \tilde{N}_p\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)} = 0.$$

2. Let $\{N_{p_n}\}_{n=1}^{\infty} \subset \mathbb{R}\mathcal{H}^{\infty}$ *denote a sequence of functions which uniformly approximate* N_p*; i.e.*

$$\lim_{n\to\infty} \|N_{p_n} - N_p\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)} = 0.$$

Construction of Approximants $\{P_n\}_{n=1}^{\infty}$

3. *Let* $\{\tilde{D}_{p_n}\}_{n=1}^{\infty} \subset \mathbb{R}\mathcal{H}^{\infty}$ *denote a sequence of functions which uniformly approximate* $\tilde{D}_p$*; i.e.*

$$\lim_{n\to\infty} \|\tilde{D}_{p_n} - \tilde{D}_p\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)} = 0.$$

4. *Let* $\{D_{p_n}\}_{n=1}^{\infty} \subset \mathbb{R}\mathcal{H}^{\infty}$ *denote a sequence of functions which uniformly approximate* D_p*; i.e.*

$$\lim_{n\to\infty} \|D_{p_n} - D_p\|_{\mathcal{H}^{\infty}(\mathbb{C}_+)} = 0.$$

5. $P_n \overset{\mathrm{def}}{=} (\tilde{D}_{p_n})^{-1}(\tilde{N}_{p_n}) = N_{p_n}(D_{p_n})^{-1}.$

Discrete-Time Plant Results from P_n Construction

1. Let $P_n^d \overset{\text{def}}{=} SFP_nH = (\tilde{D}_{p_n}^d)^{-1}(\tilde{N}_{p_n}^d) = N_{p_n}^d(D_{p_n}^d)^{-1}$ and $P^d \overset{\text{def}}{=} SFPH = (\tilde{D}_p^d)^{-1}(\tilde{N}_p^d) = N_p^d(D_p^d)^{-1}$, then $P_n^d \overset{n\to\infty}{\longrightarrow} P^d$ in the uniform norm,

$$\begin{aligned}
\sup_{\exp(j\theta),0\le\theta<2\pi} \|(\tilde{D}_{p_n}^d)^{-1}(\tilde{N}_{p_n}^d)-(\tilde{D}_p^d)^{-1}(\tilde{N}_p^d)\| &= \sup_{\exp(j\theta),0\le\theta<2\pi} \|P_n^d-P^d\| \\
&= \sup_{\exp(j\theta),0\le\theta<2\pi} \|SFP_nH-SFPH\| \\
&= \sup_{\exp(j\theta),0\le\theta<2\pi} \|SF(P_n-P)H\| \\
&\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Discrete-Time Plant Results from P_n Construction

2. Due to the above uniform convergence and the coprimeness of $\tilde{D}^d_{p_n}, \tilde{N}^d_{p_n}$ and $\tilde{D}^d_p, \tilde{N}^d_p$,

$$\lim_{n\to\infty} \|\tilde{N}^d_{p_n} - \tilde{N}^d_p\|_{\mathcal{H}^\infty(\mathbb{D})} \quad = \quad 0.$$

   Similarly, $N^d_{p_n} \overset{n\to\infty}{\longrightarrow} N^d_p, \tilde{D}^d_{p_n} \overset{n\to\infty}{\longrightarrow} \tilde{D}^d_p$ and $D^d_{p_n} \overset{n\to\infty}{\longrightarrow} D^d_p$.

3. $\tilde{D}^d_p \in \mathbb{R}\mathcal{H}^\infty(\mathbb{D})$ and $D^d_p \in \mathbb{R}\mathcal{H}^\infty(\mathbb{D})$.

Bezout Factors Assumption

Assumption 3.

$$\lim_{n\to\infty} \|N_{k_n}^d - N_k^d\|_{\mathcal{H}^\infty(\mathbb{D})} = 0$$

$$\lim_{n\to\infty} \|D_{k_n}^d - D_k^d\|_{\mathcal{H}^\infty(\mathbb{D})} = 0$$

Lifted System Parameter Properties

Lemma 2. *The lifted weighting filters*

$$\underline{\hat{W}_e}, \underline{\hat{W}_u}, \underline{\hat{W}_u}^{-1} \in \mathcal{A}.$$

The lifted coprime factor and approximants can be made to satisfy

$$\begin{aligned}
\lim_{n\to\infty} \|\underline{\hat{\tilde{N}}_{p_n}} - \underline{\hat{\tilde{N}}_p}\|_{\mathcal{H}^\infty} &= 0 \\
\lim_{n\to\infty} \|\underline{\hat{\tilde{D}}_{p_n}} - \underline{\hat{\tilde{D}}_p}\|_{\mathcal{H}^\infty} &= 0 \\
\lim_{n\to\infty} \|\underline{\hat{N}_{p_n}} - \underline{\hat{N}_p}\|_{\mathcal{H}^\infty} &= 0
\end{aligned}$$

$$
\begin{aligned}
\lim_{n\to\infty} \|\underline{\hat{D}_{p_n}} - \underline{\hat{D}_p}\|_{\boldsymbol{\mathcal{H}}^\infty} &= 0 \\
\lim_{n\to\infty} \|\underline{\hat{N}_{k_n}} - \underline{\hat{N}_k}\|_{\boldsymbol{\mathcal{H}}^\infty} &= 0 \\
\lim_{n\to\infty} \|\underline{\hat{D}_{k_n}} - \underline{\hat{D}_k}\|_{\boldsymbol{\mathcal{H}}^\infty} &= 0.
\end{aligned}
$$

Proof. This follows from Assumptions 1 and 3, Construction 1, and Proposition 1. □

Uniform Boundedness of μ_{opt} and μ_n

$$\begin{aligned}
\mu_{opt} \quad &\leq \quad \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}}\,\underline{\hat{N}_k}\,\underline{\hat{\tilde{D}}_p}\,\underline{\hat{F}}] \\ \underline{\hat{W}_u}\;\underline{\hat{N}_k}\,\underline{\hat{\tilde{D}}_p}\,\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty} < \infty \\
\mu_n \quad &\leq \quad \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}\,\underline{\hat{N}_{k_n}}\,\underline{\hat{\tilde{D}}_{p_n}}\,\underline{\hat{F}}] \\ \underline{\hat{W}_u}\;\underline{\hat{N}_{k_n}}\,\underline{\hat{\tilde{D}}_{p_n}}\,\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty}
\end{aligned}$$

for all $n \geq 1$ and

$$\left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}\,\underline{\hat{N}_{k_n}}\,\underline{\hat{\tilde{D}}_{p_n}}\,\underline{\hat{F}}] \\ \underline{\hat{W}_u}\;\underline{\hat{N}_{k_n}}\,\underline{\hat{\tilde{D}}_{p_n}}\,\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty} \quad \overset{n\rightarrow\infty}{\longrightarrow} \quad \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}}\,\underline{\hat{N}_k}\,\underline{\hat{\tilde{D}}_p}\,\underline{\hat{F}}] \\ \underline{\hat{W}_u}\;\underline{\hat{N}_k}\,\underline{\hat{\tilde{D}}_p}\,\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty}.$$

Problem Solutions: Upper-semicontinuity

Let $\mathcal{Q}(\hat{Q}_o^d) = \underline{\hat{Q}_o} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$ satisfy the following inequality:

$$\left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}}(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}} \end{array} \right\|_{\boldsymbol{\mathcal{H}}^\infty} \leq \mu_{opt} + \varepsilon \qquad (19)$$

Upper-semicontinuity

Proposition 5. *Given Assumptions 1 and 3, it follows that*

$$\mu_n \leq \mu_{opt} + 2\varepsilon \tag{20}$$

for all $n \geq N$. *Moreover,*

$$\lim_{n\to\infty} \inf_{k\geq n} \mu_k \leq \lim_{n\to\infty} \sup_{k\geq n} \mu_k \leq \mu_{opt} \tag{21}$$

Upper-semicontinuity

Proof. Consider the following inequality:

$$\mu_n \leq \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty} \tag{22}$$

$$\leq \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}}(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty} + \left\| \begin{array}{c} \underline{\hat{W}_e}[(\underline{\hat{P}} - \underline{\hat{P}_n})(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_p} + \underline{\hat{P}_n\hat{\delta}_n}]\underline{\hat{F}} \\ -\underline{\hat{W}_u}\,\underline{\hat{\delta}_n}\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty} \tag{23}$$

where

$$\underline{\hat{\delta}_n} \overset{\text{def}}{=} [(\underline{\hat{N}_k} - \underline{\hat{N}_{k_n}}) + (\underline{\hat{D}_p} - \underline{\hat{D}_{p_n}})\underline{\hat{Q}_o}]\underline{\hat{\tilde{D}}_p} + (\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_o})(\underline{\hat{\tilde{D}}_p} - \underline{\hat{\tilde{D}}_{p_n}}). \tag{24}$$

Using the near-optimality of $\underline{\hat{Q}_o}$ [see Equation (19)] yields

$$\mu_n \leq \mu_{opt} + \varepsilon + \left\| \begin{array}{c} \underline{\hat{W}_e}[(\underline{\hat{P}} - \underline{\hat{P}_n})(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_o})\underline{\hat{\tilde{D}}_p} + \underline{\hat{P}_n\hat{\delta}_n}]\underline{\hat{F}} \\ -\underline{\hat{W}_u}\,\underline{\hat{\delta}_n}\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^{\infty}} \tag{25}$$

The proof then follows from Construction 1 and Lemma 2. □

Lower-semicontinuity

Let $\mathcal{Q}(\hat{Q}_n^d) = \underline{\hat{Q}_n} \in \underline{\mathcal{H}}^\infty(\mathbb{D})$ satisfy the following inequality:

$$\left\| \begin{matrix} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}} \end{matrix} \right\|_{\mathcal{H}^\infty} \leq \mu_n + \varepsilon \qquad (26)$$

Assumption 4. [No Poles on Imaginary Axis]

$$\inf_{\omega} |D_p(j\omega)| \geq M > 0$$

Lower-semicontinuity

Proposition 6. *Given Assumptions 1 and 3, it follows that*

$$\|\underline{Q_n^F}\|_{\mathcal{H}^\infty} \leq B \tag{27}$$

for all $n \in \mathbb{Z}_+$ *and*

$$\mu_{opt} \leq \mu_n + 2\varepsilon \tag{28}$$

for all $n \geq N$. *Moreover,*

$$\mu_{opt} \leq \lim_{n\to\infty} \inf_{k\geq n} \mu_k \leq \lim_{n\to\infty} \sup_{k\geq n} \mu_k \tag{29}$$

Lower-semicontinuity

Proof. From Assumption 4, it follows that $\|\underline{\hat{D}_{p_n}}\|_{\mathcal{K}^2\to\mathcal{K}^2} \geq \|\underline{\hat{D}_p}\|_{\mathcal{K}^2\to\mathcal{K}^2} - \|\underline{\hat{D}_{p_n}} - \underline{\hat{D}_p}\|_{\mathcal{K}^2\to\mathcal{K}^2} \geq M_d - \varepsilon$, where $M_d > 0$. Evaluating $\underline{\hat{Q}_n^F}$ on the boundary of the unit circle yields

$$
\begin{aligned}
\|\underline{\hat{Q}_n^F}(e^{j\theta})\|_{\mathcal{K}^2\to\mathcal{K}^2} \quad = \quad & \|\underline{\hat{D}_{p_n}^{-1}\hat{D}_{p_n}^{-1}}\,\underline{\hat{W}_u^{-1}}(\underline{\hat{W}_u}\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}} + \underline{\hat{W}_u}\underline{\hat{D}_{p_n}\hat{D}_{p_n}}\underline{\hat{Q}_n^F} - \\
& \underline{\hat{W}_u}\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}})\|_{\mathcal{K}^2\to\mathcal{K}^2} \\
\leq \|\underline{\hat{D}_{p_n}^{-1}\hat{D}_{p_n}^{-1}}\,\underline{\hat{W}_u^{-1}}[\underline{\hat{W}_u}(\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}} + \underline{\hat{D}_{p_n}\hat{D}_{p_n}}\underline{\hat{Q}_n^F})]\|_{\mathcal{K}^2\to\mathcal{K}^2} + \\
& \|\underline{\hat{D}_{p_n}^{-1}\hat{D}_{p_n}^{-1}}\,\underline{\hat{W}_u^{-1}}(\underline{\hat{W}_u}\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}})\|_{\mathcal{K}^2\to\mathcal{K}^2} \\
\leq \|\underline{\hat{D}_{p_n}}\|_{\mathcal{K}^2\to\mathcal{K}^2}^{-2}\|\underline{\hat{W}_u^{-1}}\|_{\mathcal{K}^2\to\mathcal{K}^2}[\|\underline{\hat{W}_u}(\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}} + \underline{\hat{D}_{p_n}\hat{D}_{p_n}}\underline{\hat{Q}_n}\underline{\hat{F}})\|_{\mathcal{K}^2\to\mathcal{K}^2} \\
& \|\underline{\hat{W}_u}\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}}\|_{\mathcal{K}^2\to\mathcal{K}^2}]
\end{aligned}
$$

$$\leq \left(\frac{1}{M_d - \varepsilon}\right)^2 \|\underline{\hat{W}_u^{-1}}\|_{\mathcal{K}^2 \to \mathcal{K}^2} [\|\underline{\hat{W}_u}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{D}_{p_n}}\underline{\hat{F}}\|_{\mathcal{K}^2 \to \mathcal{K}^2} + \|\underline{\hat{W}_u}\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}}\|_{\mathcal{K}^2 \to \mathcal{K}^2}]$$

Now evaluating $\|\underline{\hat{Q}_n^F}\|_{\boldsymbol{\mathcal{H}}^\infty}$ yields

$$\begin{aligned} \|\underline{\hat{Q}_n^F}\|_{\boldsymbol{\mathcal{H}}^\infty} &= ess \sup_{\theta \in [-\pi, \pi]} \|\underline{\hat{Q}_n^F}(e^{j\theta})\|_{\mathcal{K}^2 \to \mathcal{K}^2} \\ &\leq \left(\frac{1}{M_d - \varepsilon}\right)^2 \|\underline{\hat{W}_u^{-1}}\|_{\boldsymbol{\mathcal{H}}^\infty} [\underline{\hat{W}_u}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{D}_{p_n}}\underline{\hat{F}}]\|_{\boldsymbol{\mathcal{H}}^\infty} + \\ &\quad \|\underline{\hat{W}_u}\underline{\hat{N}_{k_n}}\underline{\hat{D}_{p_n}}\underline{\hat{F}}\|_{\boldsymbol{\mathcal{H}}^\infty}] \end{aligned}$$

$$\begin{aligned}
&\leq 2\left(\frac{1}{M_d^2} + \frac{2\varepsilon/M_d^3}{1-2\varepsilon/M_d}\right)\|\underline{\hat{W}_u^{-1}}\|_{\boldsymbol{\mathcal{H}}^\infty}(\mu_n+\varepsilon)\\
&\leq \frac{2}{M_d^2}\|\underline{\hat{W}_u^{-1}}\|_{\boldsymbol{\mathcal{H}}^\infty}\mu_n + O(\varepsilon)\\
&\leq B < \infty
\end{aligned}$$

since μ_n is uniformly bounded. This demonstrates that $\underline{\hat{Q}_n}\underline{\hat{F}} = \underline{\hat{Q}_n^F} \in \boldsymbol{\mathcal{H}}^\infty$ for all $n \geq 1$. Consider the following inequality:

$$\begin{aligned}
\mu_{opt} &\leq \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}}(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_p}\underline{\hat{F}} \end{array} \right\|_{\boldsymbol{\mathcal{H}}^\infty}\\
&\leq \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}} \end{array} \right\|_{\boldsymbol{\mathcal{H}}^\infty} +
\end{aligned}$$

$$\left\| \begin{matrix} -\underline{\hat{W}_e}[(\underline{\hat{P}} - \underline{\hat{P}_n})(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_p} + \underline{\hat{P}_n}\underline{\hat{\zeta}_n}]\underline{\hat{F}} \\ \underline{\hat{W}_u}\,\underline{\hat{\zeta}_n}\underline{\hat{F}} \end{matrix} \right\|_{\mathcal{H}^\infty} \tag{30}$$

where

$$\underline{\hat{\zeta}_n} \overset{\text{def}}{=} [(\underline{\hat{N}_k} - \underline{\hat{N}_{k_n}}) + (\underline{\hat{D}_p} - \underline{\hat{D}_{p_n}})\underline{\hat{Q}_n}]\underline{\hat{\tilde{D}}_p} + (\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})(\underline{\hat{\tilde{D}}_p} - \underline{\hat{\tilde{D}}_{p_n}}). \tag{31}$$

Using the near-optimality of $\underline{\hat{Q}_n}$ [see Equation (26)] yields

$$\mu_{opt} \leq \mu_n + \varepsilon + \left\| \begin{matrix} -\underline{\hat{W}_e}[(\underline{\hat{P}} - \underline{\hat{P}_n})(\underline{\hat{N}_k} + \underline{\hat{D}_p}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_p} + \underline{\hat{P}_n}\underline{\hat{\zeta}_n}]\underline{\hat{F}} \\ \underline{\hat{W}_u}\,\underline{\hat{\zeta}_n}\underline{\hat{F}} \end{matrix} \right\|_{\mathcal{H}^\infty} \tag{32}$$

The proof then follows from Construction 1 and Lemma 2. □

Solution to Purely Finite-Dimensional Problem

Theorem 3. *Given Assumptions 1 and 3, it follows that*

$$|\mu_n - \mu_{opt}| \leq 2\varepsilon \tag{33}$$

Moreover,

$$\lim_{n\to\infty} \mu_n = \mu_{opt} \tag{34}$$

Actual Performance Measure $\tilde{\mu}_n$

$$\begin{aligned}
\tilde{\mu}_n &= \left\| \begin{pmatrix} \underline{\hat{W}_e} \\ \underline{\hat{W}_u}\,\underline{\hat{K}_n} \end{pmatrix} \left(I - \underline{\hat{P}}\,\underline{\hat{K}_n} \right)^{-1} \right\|_{\mathcal{H}^\infty} \\
&= \left\| \begin{matrix} \underline{\hat{W}_e}[I - \underline{\hat{P}}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\delta}^{-1}\underline{\hat{\tilde{D}}_p}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\delta}^{-1}\underline{\hat{\tilde{D}}_p}\underline{\hat{F}} \end{matrix} \right\|_{\mathcal{H}^\infty} \qquad (35)
\end{aligned}$$

where

$$\underline{\delta} = \underline{\hat{\tilde{D}}_p}(\underline{\hat{D}_{k_n}} - \underline{\hat{N}_{p_n}}\underline{\hat{Q}_n}) + \underline{\hat{\tilde{N}}_p}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})$$

and for all $n \geq N$.

Stability of Actual Closed Loop Operator: $(\underline{\hat{P}}, \underline{\hat{K}_n})$

Proposition 7. *$\underline{\delta}$ is invertible in $\mathcal{H}^\infty$ and $\underline{\hat{K}_n}$ internally stabilizes $\underline{\hat{P}}$ for all $n > N$.*

Stability of Actual Closed Loop Operator: $(\underline{\hat{P}}, \underline{\hat{K}_n})$

Proof.

$$\lim_{n\to\infty} \|I - \underline{\delta}\|_{\mathcal{H}^\infty} \quad = \quad \lim_{n\to\infty} \|I - [\underline{\hat{\tilde{D}}_p}(\underline{\hat{D}_{k_n}} - \underline{\hat{N}_{p_n}}\underline{\hat{Q}_n}) + \underline{\hat{\tilde{N}}_p}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})]\|_{\mathcal{H}}$$

$$= \lim_{n\to\infty} \|I - [\underline{\hat{\tilde{D}}_p}\underline{\hat{D}_k} - \underline{\hat{\tilde{D}}_p}\underline{\hat{D}_k} + \underline{\hat{N}_p}\underline{\hat{N}_k} - \underline{\hat{N}_p}\underline{\hat{N}_k} + \underline{\hat{\tilde{D}}_p}(\underline{\hat{D}_{k_n}} - \underline{\hat{N}_{p_n}}\underline{\hat{Q}_n}) +$$

$$\underline{\hat{\tilde{N}}_p}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})]\|_{\mathcal{H}^\infty}$$

$$= \lim_{n\to\infty} \|\underline{\hat{\tilde{D}}_p}\underline{\hat{D}_k} + \underline{\hat{N}_p}\underline{\hat{N}_k} - \underline{\hat{\tilde{D}}_p}(\underline{\hat{D}_{k_n}} - \underline{\hat{N}_{p_n}}\underline{\hat{Q}_n}) - \underline{\hat{\tilde{N}}_p}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\|_{\mathcal{H}^\infty}$$

$$= \lim_{n\to\infty} \|\underline{\hat{\tilde{D}}_p}\underline{\hat{D}_k} + \underline{\hat{N}_p}\underline{\hat{N}_k} - \underline{\hat{\tilde{D}}_p\hat{D}_{k_n}} + \underline{\hat{\tilde{D}}_p\hat{N}_{p_n}}\underline{\hat{Q}_n} - \underline{\hat{\tilde{N}}_p\hat{N}_{k_n}} - \underline{\hat{\tilde{N}}_p\hat{D}_{p_n}}\underline{\hat{Q}_n}\|_{\mathcal{H}^\infty}$$

$$= \lim_{n\to\infty} \|\underline{\hat{\tilde{D}}_p}(\underline{\hat{D}_k} - \underline{\hat{D}_{k_n}}) + \underline{\hat{\tilde{N}}_p}(\underline{\hat{N}_k} - \underline{\hat{N}_{k_n}}) + \underline{\hat{\tilde{D}}_p\hat{N}_{p_n}}\underline{\hat{Q}_n} - \underline{\hat{\tilde{N}}_p\hat{D}_{p_n}}\underline{\hat{Q}_n}\|_{\mathcal{H}^\infty}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \|\underline{\hat{\tilde{D}}_p}(\underline{\hat{D}_k} - \underline{\hat{D}_{k_n}}) + \underline{\hat{\tilde{N}}_p}(\underline{\hat{N}_k} - \underline{\hat{N}_{k_n}}) + (\underline{\hat{\tilde{D}}_p \hat{N}_{p_n}} - \underline{\hat{\tilde{N}}_p \hat{D}_{p_n}})\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^\infty} \\
&= \lim_{n\to\infty} \|(\underline{\hat{\tilde{D}}_p \hat{N}_{p_n}} - \underline{\hat{\tilde{N}}_p \hat{D}_{p_n}})\underline{\hat{Q}_n}\|_{\boldsymbol{\mathcal{H}}^\infty} \\
&= 0
\end{aligned}
$$

using the fact that $\underline{\hat{Q}_n}$ is uniformly bounded since $\underline{\hat{Q}_n^F}$ is uniformly bounded and there is no "unstable" pole-zero type cancellations between $\underline{\hat{Q}_n}$ and $\underline{\hat{F}}$. This follows from

$$
\begin{aligned}
B &\geq \|\underline{\hat{Q}_n}\underline{\hat{F}}\|_{\boldsymbol{\mathcal{H}}^\infty} \\
&= \|\underline{Q_n}\underline{F}\|_{\boldsymbol{\ell}^2\to\boldsymbol{\ell}^2} \\
&= \|LHQ_n^d SFL^{-1}\|_{\boldsymbol{\ell}^2\to\boldsymbol{\ell}^2} \\
&= \|(SFL^{-1})^* Q_n^{d*}(LH)^*(LH)Q_n^d(SFL^{-1})\|_{\boldsymbol{\ell}^2\to\boldsymbol{\ell}^2}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \|(SFL^{-1})^* Q_n^{d*} M_H^{d*} M_H^d Q_n^d (SFL^{-1})\|_{\ell^2 \to \ell^2}^{\frac{1}{2}} \\
&= \|M_H^d Q_n^d (SFL^{-1})\|_{\ell^2 \to \ell^2} \\
&= \|M_H^d Q_n^d (SFL^{-1})(SFL^{-1})^* Q_n^{d*} M_H^{d*}\|_{\ell^2 \to \ell^2}^{\frac{1}{2}} \\
&= \|M_H^d Q_n^d M_{SF}^d M_{SF}^{d*} Q_n^{d*} M_H^{d*}\|_{\ell^2 \to \ell^2}^{\frac{1}{2}} \\
&= \|M_H^d Q_n^d M_{SF}^d\|_{\ell^2 \to \ell^2} \\
&= \|M_H^d Q_n^d M_{SF}^d\|_{\mathcal{H}^\infty(\mathbb{D})}
\end{aligned}
$$

where * represents operator adjoint and M_H^d, M_{SF}^d are stable finite-dimensional linear shift-invariant systems associated with the operators LH, SFL^{-1}, respectively, such that

$$
\|LH\|_{\ell^2 \to \ell^2} = \|M_H^d\|_{\ell^2 \to \ell^2}
$$

$$\|SFL^{-1}\|_{\ell^2\to\ell^2} \quad = \quad \|M^d_{SF}\|_{\ell^2\to\ell^2}.$$

Note that $M^d_H = \sqrt{h}$. Evaluating $|Q^d_n|$ on the unit circle under the assumption that M^d_{SF} has no zeros on the unit circle results in

$$\begin{aligned} |Q^d_n| &= |M^{-d}_H M^d_H Q^d_n M^d_{SF} M^{-d}_{SF}| \\ &\leq \frac{1}{\sqrt{h}} B |M^{-d}_{SF}| \\ &< B' < \infty. \end{aligned}$$

This implies that

$$\|Q^d_n\|_{\mathcal{H}^\infty(\mathbb{D})} \quad < \quad B'.$$

Since Q^d_n is uniformly bounded in the $\mathcal{H}^\infty(\mathbb{D})$ norm, it converges to an element in $\mathcal{H}^\infty(\mathbb{D})$. Thus, $\underline{\hat{Q}_n}$ is uniformly bounded in the $\boldsymbol{\mathcal{H}}^\infty$ norm and converges to an element in $\boldsymbol{\mathcal{H}}^\infty$.

To prove that $\underline{\hat{K}_n}$ internally stabilizes $\underline{\hat{P}}$ for n sufficiently large, one argues as follows. Since $\underline{\hat{K}_n}$ internally stabilizes $\underline{\hat{P}_n}$, it follows that $\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n}$ and $\underline{\hat{D}_{k_n}} - \underline{\hat{N}_{p_n}}\underline{\hat{Q}_n}$ must be coprime in $\boldsymbol{\mathcal{H}}^\infty$. Note that $\underline{\hat{N}_{p_n}}$ and $\underline{\hat{D}_{p_n}}$ are also coprime in $\boldsymbol{\mathcal{H}}^\infty$. Consequently, it follows that $\underline{\hat{K}_n}$ internally stabilizes $\underline{\hat{P}}$ if and only if $\underline{\delta}$ is a unit of (i.e. invertible in) $\boldsymbol{\mathcal{H}}^\infty$. However, the previous result implies that $\underline{\delta}$ will be a unit for all but a finite number of n. $\square$

M_{SF}^d Zero Assumption

Comment. *The assumption that* M_{SF}^d *has no zeros on the unit circle isn't restrictive in that an AAF of the form* $\frac{a}{s+a}$ *results in*

$$\|SFL^{-1}\|_{\ell^2 \to \ell^2} = \left\| \left[\frac{a(1-e^{-2ah})}{2} \right]^{\frac{1}{2}} \frac{z}{1-e^{-ah}z} \right\|_{\mathcal{H}^\infty(\mathbb{D})}$$

which is seen not to have a zero on the unit circle.

Solution to $\mathcal{H}^\infty$ Approximate/Design Mixed-Sensitivity Problem

Theorem 4. *Given Assumptions 1 and 3, it follows that*

$$\mu_{opt} \leq \tilde{\mu}_n \leq \mu_{opt} + 4\varepsilon \tag{36}$$

for all $n \geq N$. *Moreover,*

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}. \tag{37}$$

Solution to $\mathcal{H}^\infty$ Approximate/Design Mixed-Sensitivity Problem

Proof. From Equation (35), one obtains the following inequality:

$$
\begin{aligned}
\tilde{\mu}_n \quad &= \quad \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\delta}^{-1}\underline{\hat{\tilde{D}}_p}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\delta}^{-1}\underline{\hat{\tilde{D}}_p}\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty} \\
&\leq \quad \left\| \begin{array}{c} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty} +
\end{aligned}
$$

$$
\left\| \begin{array}{c} \underline{\hat{W}_e}[(\underline{\hat{P}_n} - \underline{\hat{P}})(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\delta}^{-1}\underline{\hat{\tilde{D}}_p} + \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})(\underline{\hat{D}_{p_n}} - \underline{\hat{\tilde{D}}_p}) + \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})(I - \underline{\delta}^{-1})\underline{\hat{\tilde{D}}_p}]\underline{\hat{F}} \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})[\underline{\delta}^{-1}(\underline{\hat{\tilde{D}}_p} - \underline{\hat{D}_{p_n}}) - (I - \underline{\delta}^{-1})\underline{\hat{\tilde{D}}_{p_n}}]\underline{\hat{F}} \end{array} \right\|_{\mathcal{H}^\infty}
$$

Since $\underline{\hat{K}_n} \overset{\text{def}}{=} (\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})(\underline{\hat{D}_{k_n}} - \underline{\hat{N}_{p_n}}\underline{\hat{Q}_n})^{-1}\underline{\hat{F}}$ stabilizes $\underline{\hat{P}}$ for all but a finite number of n, it follows that

$$\mu_{opt} \leq \tilde{\mu}_n \tag{38}$$

for all $n \geq N$. Since $\underline{\hat{Q}_n}$ satisfies the inequality (26), it follows from Theorem 3 that

$$\begin{aligned} \left\| \begin{matrix} \underline{\hat{W}_e}[I - \underline{\hat{P}_n}(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}}] \\ \underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}} + \underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\hat{\tilde{D}}_{p_n}}\underline{\hat{F}} \end{matrix} \right\|_{\mathcal{H}^\infty} &\leq \mu_n + \varepsilon \\ &\leq \mu_{opt} + 3\varepsilon \end{aligned}$$

for all $n \geq N$. This, then yields

$$\begin{aligned}\mu_{opt} &\leq \tilde{\mu}_n \\ &\leq \mu_{opt} + 3\varepsilon + \end{aligned}$$

$$\left\| \frac{\underline{\hat{W}_e}[(\underline{\hat{P}_n}-\underline{\hat{P}})(\underline{\hat{N}_{k_n}}+\underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})\underline{\delta}^{-1}\underline{\hat{\tilde{D}}_p}+\underline{\hat{P}_n}(\underline{\hat{N}_{k_n}}+\underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})(\underline{\hat{D}_{p_n}}-\underline{\hat{\tilde{D}}_p})+\underline{\hat{P}_n}(\underline{\hat{N}_{k_n}}+\underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})(I-\underline{\delta}^{-1})\underline{\hat{\tilde{D}}_p}]\underline{\hat{F}}}{\underline{\hat{W}_u}\,(\underline{\hat{N}_{k_n}}+\underline{\hat{D}_{p_n}}\underline{\hat{Q}_n})[\underline{\delta}^{-1}(\underline{\hat{\tilde{D}}_p}-\underline{\hat{D}_{p_n}})-(I-\underline{\delta}^{-1})\underline{\hat{\tilde{D}}_{p_n}}]\underline{\hat{F}}} \right\|_{\mathcal{H}}$$

for all $n \geq N$. Using the uniform bound for $\underline{\hat{Q}_n^F}$ obtained in (27) yields

$$\mu_{opt} \leq \tilde{\mu}_n \leq \mu_{opt} + 4\varepsilon \qquad (39)$$

for all $n \geq N$. The proof then follows from Construction 1 and Lemma 2. □

Design Methodology

1. Start with the sampled-data system with infinite-dimensional plant, P, and performance measure, μ_{opt}, which takes into account intersample behavior. A specified performance criterion is that the actual performance be near-optimal. The actual performance is defined as the performance achieved from the sampled-data system using the infinite-dimensional plant and a finite-dimensional discrete-time controller, K_n^d.

2. Approximate the infinite-dimensional plant coprime factors with finite-dimensional $\mathbb{R}\mathcal{H}^\infty(\mathbb{C}_+)$ approximants. For the stable plant case, the approximant order has been shown to be determinable *a priori* based on ε_d and the weighting filters. Use these approximants in place of the infinite-dimensional plant in the sampled-data set-up.

3. Lift the resultant finite-dimensional sampled-data system (Figure 2).

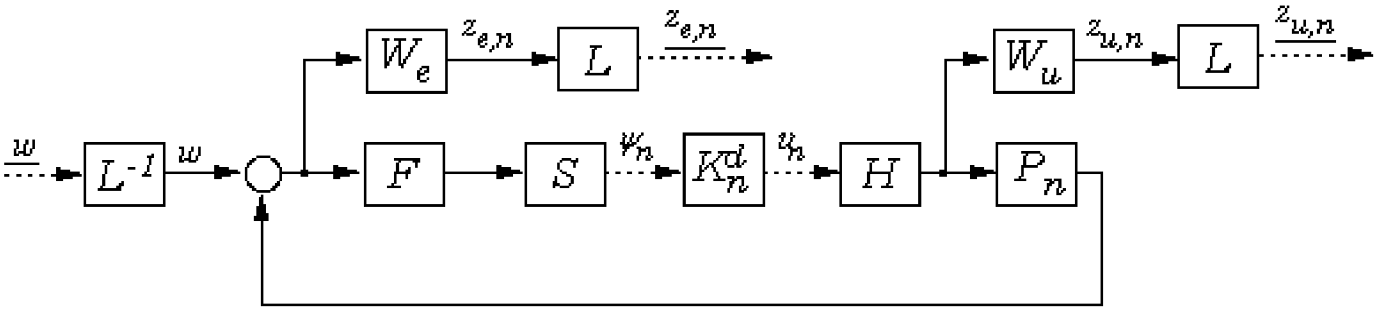


Figure 2: Sampled-Data Feedback Loop Input/Output Lifting

4. In Figure 2, $\mathcal{H}^\infty$-discretize the operator which maps $\left[\begin{array}{c} \underline{w} \\ v_n \end{array}\right] \rightarrow \left[\begin{array}{c} \underline{z_n} \\ \psi_n \end{array}\right]$.

5. Synthesize a finite-dimensional discrete-time controller, K_n^d, using "natural" discrete-time $\mathcal{H}^\infty(\mathbb{D})$ design algorithms based on the discretized finite-dimensional operator.

6. The synthesized finite-dimensional discrete-time controller results in stable closed loop performance for the original infinite-dimensional sampled-data system with guaranteed performance measure satisfying $\mu_{opt} \leq \tilde{\mu}_n \leq \mu_{opt} + 4\varepsilon$.

Unstable Delay Numerical Example

Infinite-Dimensional Plant

$$\hat{P}(s) = \frac{e^{-s}}{s - 0.01}.$$

Sample Rate

$$T_s = 0.2$$

Anti-Aliasing Filter

$$\hat{F}(s) = \frac{1}{\frac{0.2}{\pi}s + 1}$$

Weighting Filters

Error and Control Weighting Filters

$$
\begin{aligned}
\hat{W}_e(s) &= \frac{1}{\left(\frac{5}{2\pi}s+1\right)^2} \\
\hat{W}_u(s) &= \frac{\left(\frac{5}{2\pi}s+10^{-3}\right)}{\left(\frac{\pi}{75}s+1\right)}.
\end{aligned}
$$

Weighting Filter $\mathcal{H}^\infty$ Norms

$$
\begin{aligned}
\|\hat{W}_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} &= 1 \\
\|\hat{W}_u\|_{\mathcal{H}^\infty(\mathbb{C}_+)} &= 19
\end{aligned}
$$

Plant Approximants

Uniform Plant Approximants

$$\hat{P}_n(s) = \frac{\hat{N}_{\mathrm{p}n}(s)}{\hat{D}_{\mathrm{p}n}(s)}\left(\frac{1}{s-0.01}\right)$$

where

$$\hat{D}_{\mathrm{p}n}(s) = \sum_{k=0}^{n} \frac{(2n-k)!n!}{2n!k!(n-k)!} s^k$$

and

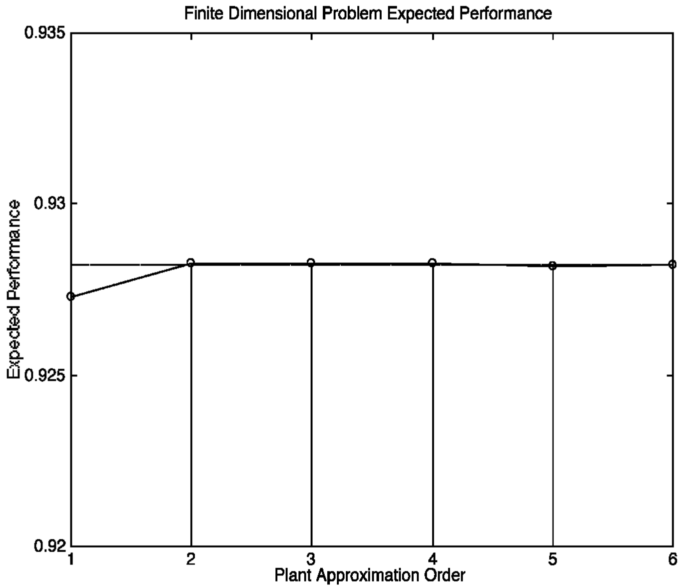
$$\hat{N}_{\mathrm{p}n}(s) = \hat{D}_{\mathrm{p}n}(-s).$$

$\hat{P}_n$ uniformly approximates $\hat{P}$

$$\lim_{n\to\infty} \sup_{\omega} \|\hat{P}_n - \hat{P}\| = 0$$

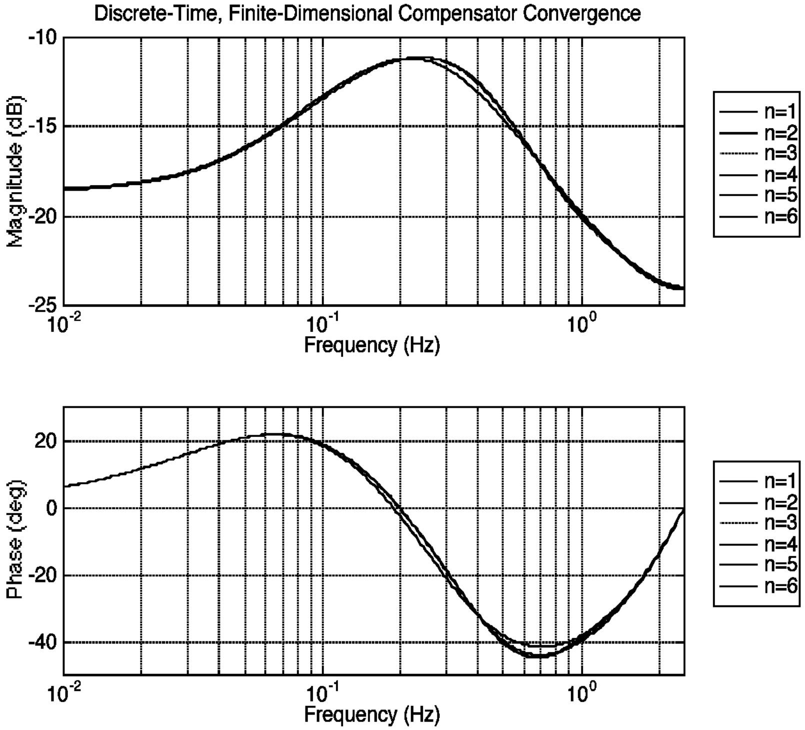
μ_{opt} Estimation

- Convergence of expected performance, μ_n



- Indicates that infinite-dimensional sampled-data system optimal performance μ_{opt} is around 0.928

Discrete-Time, Finite-Dimensional Controller Convergence



"Unstable" Diffusion Process Numerical Example

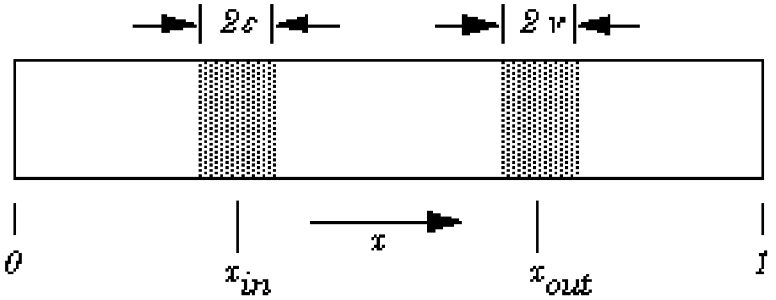


Figure 3: SDOF Heat Flow Model

Infinite-Dimensional Plant

$$
\begin{aligned}
P(s) &= \frac{1}{s+\beta} + \sum_{m=1}^{\infty} \frac{2\cos(m\pi x_{in})\sin(m\pi\varepsilon)\cos(m\pi x_{out})\sin(m\pi\nu)}{\varepsilon\nu\,(m\pi)^2\left(s+\beta+\alpha\,(m\pi)^2\right)} \\
&= P_u(s) + P_s(s)
\end{aligned}
$$

Sample Rate

$$T_s = 0.3$$

Anti-Aliasing Filter

$$\hat{F}(s) = \frac{1}{\frac{0.3}{\pi}s + 1}$$

Weighting Filters

Error and Control Weighting Filters

$$\begin{aligned} W_e(s) &= \frac{1}{\left(\frac{5}{2\pi}s+1\right)^2} \\ W_u(s) &= \frac{\frac{5}{2\pi}\left(s+10^{-3}\right)\left(\frac{s}{5\omega_s}+1\right)}{\left(\frac{1}{12\pi}s+1\right)\left(\frac{1}{100}s+1\right)} \end{aligned}$$

Weighting Filter $\mathcal{H}^\infty$ Norms

$$\begin{aligned} \|W_e\|_{\mathcal{H}^\infty(\mathbb{C}_+)} &= 1 \\ \|W_u\|_{\mathcal{H}^\infty(\mathbb{C}_+)} &= 21.7866 \end{aligned}$$

Plant Approximants

Uniform Plant Approximants

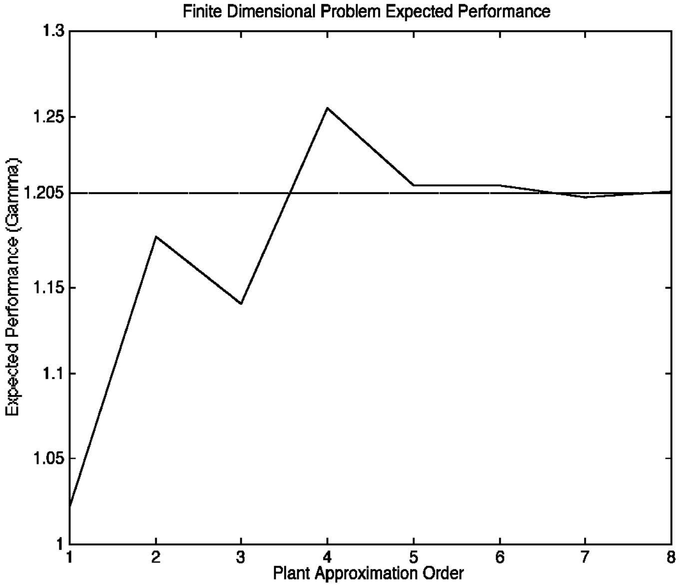
$$
\begin{aligned}
P_n(s) &= \frac{1}{s+\beta} + \sum_{m=1}^{n} \frac{2\cos(m\pi x_{in})\sin(m\pi\varepsilon)\cos(m\pi x_{out})\sin(m\pi\nu)}{\varepsilon\nu\,(m\pi)^2\left(s+\beta+\alpha\,(m\pi)^2\right)} \\
&= P_u(s) + P_{s_n}(s)
\end{aligned}
$$

P_n **uniformly approximates** P

$$\lim_{n\to\infty} \|P - P_n\|_{\mathcal{H}^\infty(\mathbb{C}_+)} = 0$$

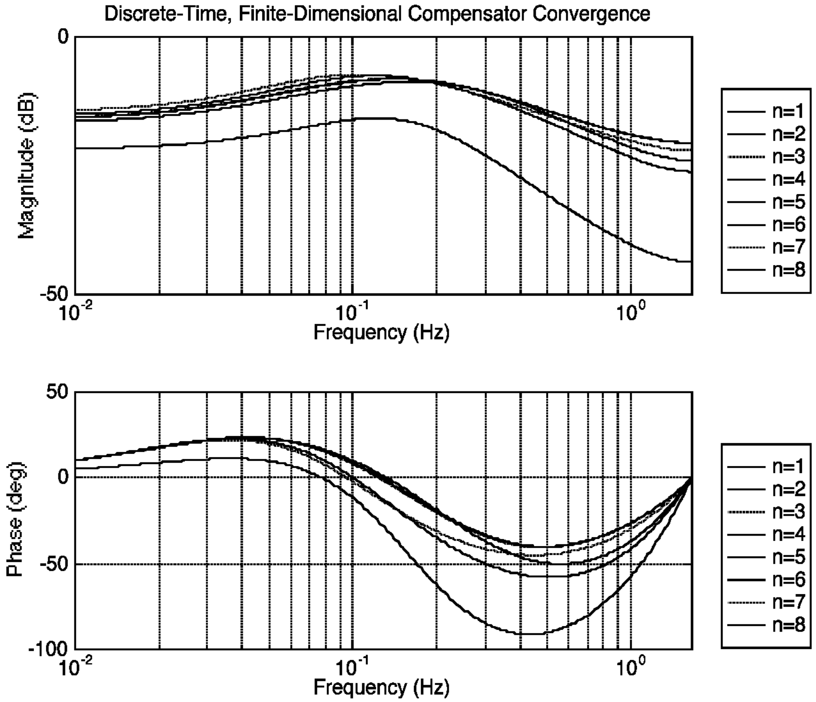
μ_{opt} **Estimation**

- Convergence of expected performance, μ_n



- Indicates that infinite-dimensional sampled-data system optimal performance μ_{opt} is around 1.205

Discrete-Time, Finite-Dimensional Controller Convergence



Summary

- Systematic methodology for synthesizing near-optimal finite-dimensional sampled-data controllers for a large class of continuous-time distributed parameter plants, based on finite-dimensional plant approximants

- For stable systems, near optimal finite-dimensional sampled-data controller synthesis can be performed by solving a single (*a priori* determinable) finite-dimensional optimal sampled-data synthesis problem

- Criteria used to determine optimality is a weighted induced $\mathcal{L}^2$ mixed-sensitivity measure which penalizes both the sensitivity operator and a operator associated with the control

- Key technical requirements are that uniform plant approximants are available and that the control is penalized in a nonsingular manner

- Optimal performance can be approximated by solving a sequence of finite dimensional sampled data problems rather than a possibly infinite-dimensional eigenvalue/eigenfunction problem

- For stable systems, optimal performance can be approximated to any arbitrary accuracy by solving a single (*a priori* determinable) finite-dimensional optimal sampled-data problem rather than a possibly infinite-dimensional eigenvalue/eigenfunction problem

Future Research

- Extensions to MIMO unstable systems
- Controller frequency response convergence results
- Generalize discretized AAF zero requirements